The AcroTEX Web Site, 1999

A Slide Show Demonstrating Newton’s Method

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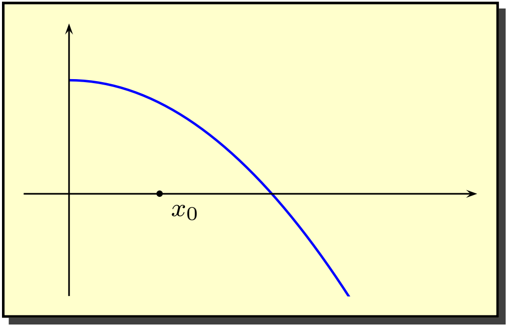
1. Newton's Method

Problem: Given an equation $f(x) = 0$, solve for x numerically.



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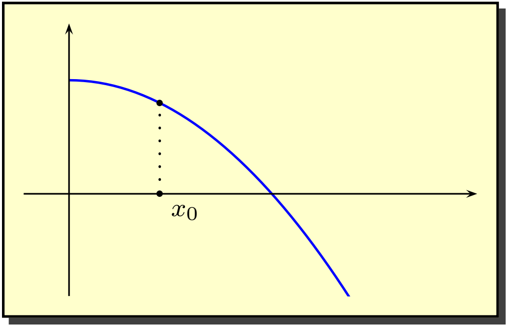

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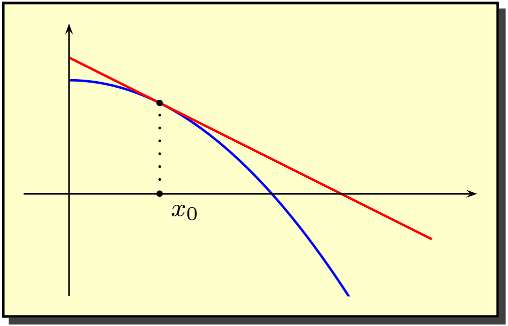

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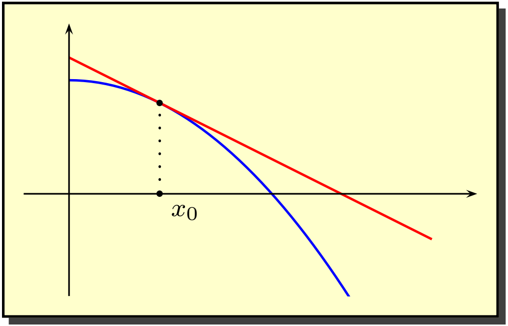

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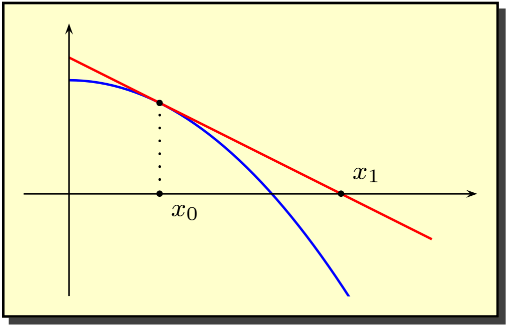

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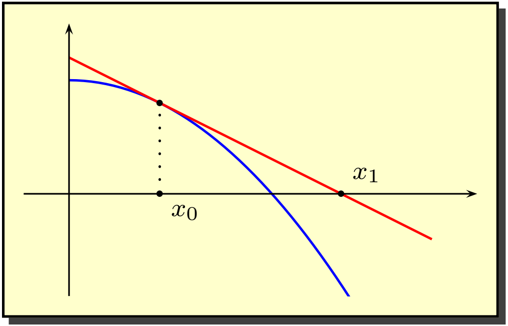

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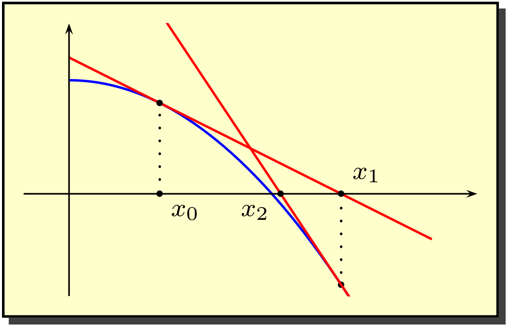

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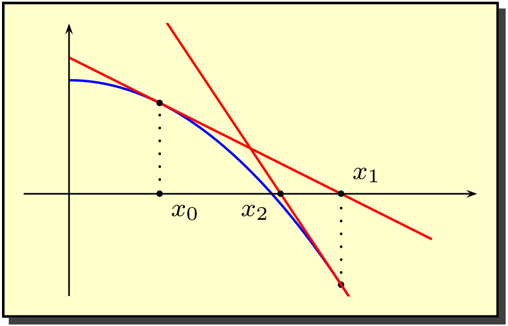

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- Let x_1 be in x-intercept of this tangent line.

- This intercept is given by the formula: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.
- Now repeat using x_1 as the initial guess.
- The intercept x_2 is given by: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

2. Commentary

The initial guess, x_0, was close to the true root. From the picture, Frame 5, it appears our next estimate x_1,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a little closer to the unknown root than x_0 was.

The next “iterate”, x_2, calculated from the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

is closer still to the unknown root, see Frame 7.

The process continues: Given that an estimate x_n has already been calculated, the equation of the tangent line is calculated:

$$y = f(x_n) + f'(x_n)(x - x_n)$$

The x-intercept is then calculated,

$$f(x_n) + f'(x_n)(x - x_n) = 0 \implies x = x_n - \frac{f(x_n)}{f'(x_n)}$$

This intercept is labeled x_{n+1} and represents our next estimate of the unknown root.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \tag{1}$$

The initial guess x_0, and the Newton Iteration formula, equation (1), together form an *algorithm* or a procedure of estimating the value of the root to the equation $f(x) = 0$.

3. Examples

Example 3.1. Find the positive root of the equation $x^2 = 2$.

Solution: The function is $f(x) = x^2 - 2$.

Step 1: Compute derivative, $f'(x) = 2x$.

Step 2: Construct the iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}$$

Thus, for this problem, the iteration formula is

$$\boxed{x_{n+1} = \frac{x_n^2 + 2}{2x_n}}$$

This, together with an initial guess of $x_0 = 1.5$ yields the following calculations.

Step 3: Construct a table of estimates.

Initial guess of $x_0 = 1.5$ and iteration formula of $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$.

Newton's Method		
$f(x) = x^2 - 2, \quad x_0 = 1.5$		
n	x_n	$f(x_n)$
0	1.50000000	0.25000000
1	1.41666667	0.00694445
2	1.41421568	0.00000600
3	1.41421356	0.00000000
4	1.41421356	0.00000000

Thus, the positive root to the equation $x^2 - 2 = 0$ is $\boxed{x \approx 1.4142135}$, or, stated differently, $\boxed{\sqrt{2} \approx 1.4142135}$.

Example 3.1. ■

Example 3.2. Find a solution to the equation $x^3 = x + 1$ that is near $x_0 = 1.5$.

Solution: The function is $f(x) = x^3 - x - 1$. The function f is *always* defined to make the given equation equivalent to an equation of the form $f(x) = 0$. (We just take everything on the right-hand side of the given equation to the left-hand side. The left-hand side is now an expression defining $f(x)$.

Step 1: Differentiate $f'(x) = 3x^2 - 1$.

Step 2: Construct the Newton iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

The iteration formula is

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}}$$

Step 3: Construct the table of estimates.

Newton's Method		
$f(x) = x^3 - x - 1, \quad x_0 = 1.5$		
n	x_n	$f(x_n)$
0	1.50000000	0.87500000
1	1.34782608	0.10058217
2	1.32520039	0.00205836
3	1.32471817	0.00000092
4	1.32471795	0.00000000
5	1.32471795	0.00000000

Thus, the solution to the equation $x^3 = x + 1$ that is "near" $x_0 = 1.5$ is $\boxed{x \approx 1.32471795}$.

Example 3.2. ■

AcroTEX: http://www.math.uakron.edu/~dpstory/acrotex.html

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